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Equivalence of the Ampère and Biot–Savart force laws in magnetostatics

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Abstract. The equivalence of the Ampère and Biot–Savart force laws in magnetostatics is examined by considering the difference in the forces predicted by the two laws. The conditions under which this force difference vanishes are discussed, with special reference to the question of convergence of the integrals involved. The conclusions are that a complete current distribution exerts forces, which are the same according to both laws, when acting on a volume current element or a macroscopic part of a current distribution, irrespective of whether these are situated outside or inside the distribution exerting the force. These forces are shown to be normal to the current density vector at all points according to both force laws. Both the force distributions and stresses are predicted equal by both laws, thus proving the impossibility of devising an experiment which might differentiate between the two laws by measurements of forces exerted on a part of a circuit by the circuit itself. The equivalence of the two force laws in magnetostatistics is considered to be complete.

1. Introduction

The equivalence of the Ampère and Biot–Savart force laws in magnetostatics has been the subject of discussion recently (Graneau 1982, 1985a, b, Ternan 1985a, b, Jolly 1985, Pappas and Moyssides 1985). Claims were made that the Ampère law provided the correct theoretical interpretation to some experiments while the Biot–Savart law failed to do so (Graneau 1982). Theoretical proof that this could not be the case since the two laws are equivalent in problems met in magnetostatics was provided both by Ternan (1985a) and Jolly (1985).

Ternan (1985a) demonstrated that both laws predict equal and opposite forces between two complementary parts of a complete current distribution and derived the Ampère law starting from the Biot–Savart law, thus showing that the two laws are equivalent. Jolly (1985) argued that both laws are special cases of a more general expression given by Whittaker (1910) and also showed how either law may be derived from the other.

There followed objections to these assertions of equivalence of the two laws based on claims that, although the total forces were predicted equal, the two laws differed in the stresses produced in conductors (Graneau 1985a, b) or, what is the same thing, in the distribution of forces (Pappas and Moyssides 1985).

It is our opinion that the proofs of equivalence both by Ternan and by Jolly are mathematically sound and any disagreement between experiment and theory either casts doubts on both laws or, more probably, on the correct application of theory to

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the particular experiment. Nevertheless it is felt that a different approach to the problem may be of benefit in clarifying matters. This is the purpose of this work which, examining the expression for the difference of the forces predicted by the two laws, shows under what circumstances this difference vanishes, thus proving the equivalence of the laws both as regards macroscopic forces and their distribution and direction, paying special attention to the question of convergence of the integrals involved in calculating the forces.

2. Mathematical analysis

Consider a steady current distribution of current density $\boldsymbol{J}(\boldsymbol{r})$, occupying volume V bounded by the closed surface S. This is shown in figure 1, where part of the current loop has been drawn using broken lines, for reasons to be explained below. At the origin O of the coordinate system where the current density is $\boldsymbol{J}'$, there is a current element of volume $\mathrm{d}v'$. To simplify notation, the latter may be considered to consist of charges q_i with velocities $\boldsymbol{u}_i$ and define $Q\boldsymbol{u}=\boldsymbol{J}'\,\mathrm{d}v'=\Sigma q_i\boldsymbol{u}_i$. A volume element $\mathrm{d}v$ is situated at $\boldsymbol{r}$, inside the current distribution where the current density is $\boldsymbol{J}$.

The Ampère force from $\mathrm{d}v$ on Q is

$$\mathrm{d}\boldsymbol{f}_{\mathrm{A}}=\frac{\mu_0 Q\,\mathrm{d}v}{4\pi r^3}\,\boldsymbol{r}\left(2(\boldsymbol{u}\cdot\boldsymbol{J})-\frac{3}{r^2}(\boldsymbol{u}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{J})\right) \tag{1}$$

and the Biot–Savart force is

$$\mathrm{d}\boldsymbol{f}_{\mathrm{BS}}=\frac{\mu_0 Q\,\mathrm{d}v}{4\pi r^3}\{\boldsymbol{u}\times[\boldsymbol{J}\times(-\boldsymbol{r})]\}$$

$$=\frac{\mu_0 Q\,\mathrm{d}v}{4\pi r^3}[\boldsymbol{r}(\boldsymbol{u}\cdot\boldsymbol{J})-\boldsymbol{J}(\boldsymbol{u}\cdot\boldsymbol{r})]. \tag{2}$$

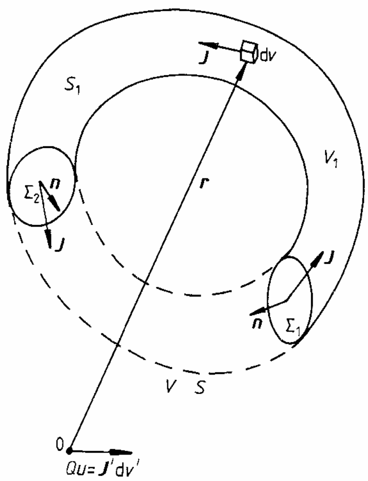

Figure 1. The geometry of the current distribution $\boldsymbol{J}$ and the volume current element $\boldsymbol{J}'\,\mathrm{d}v'$ at the origin.

Their difference is

$$\mathrm{d}\boldsymbol{f} = \mathrm{d}\boldsymbol{f}_{\mathrm{A}} - \mathrm{d}\boldsymbol{f}_{\mathrm{BS}}$$

$$= \frac{\mu_0 Q\,\mathrm{d}v}{4\pi r^3}\left(\boldsymbol{r}(\boldsymbol{u}\cdot\boldsymbol{J}) + \boldsymbol{J}(\boldsymbol{u}\cdot\boldsymbol{r}) - 3\frac{\boldsymbol{r}}{r^2}(\boldsymbol{u}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{J})\right). \qquad (3)$$

Using $\nabla\times\boldsymbol{r}=0$ and $\nabla\cdot\boldsymbol{J}=0$ we find

$$\nabla\cdot\left(\frac{\boldsymbol{J}}{r^3}(\boldsymbol{u}\cdot\boldsymbol{r})\right) = \frac{\boldsymbol{u}\cdot\boldsymbol{J}}{r^3} - \frac{3}{r^5}(\boldsymbol{u}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{J}) \qquad (4)$$

so that

$$\mathrm{d}\boldsymbol{f} = \frac{\mu_0 Q\,\mathrm{d}v}{4\pi}[\boldsymbol{N} + \boldsymbol{r}(\nabla\cdot\boldsymbol{N})] \qquad (5)$$

with

$$\boldsymbol{N} = (\boldsymbol{J}/r^3)(\boldsymbol{u}\cdot\boldsymbol{r}). \qquad (6)$$

The difference of the forces exerted on Q by a volume V_1 of the current distribution as predicted by the two laws is found by integrating $\mathrm{d}\boldsymbol{f}$ over V_1. This may be converted to a surface integral using the integral identity

$$\oint_{S_1} \boldsymbol{r}(\boldsymbol{N}\cdot\boldsymbol{n})\,\mathrm{d}s = \int_{V_1}[(\boldsymbol{N}\cdot\nabla)\boldsymbol{r} + \boldsymbol{r}(\nabla\cdot\boldsymbol{N})]\,\mathrm{d}v$$

$$= \int_{V_1}[\boldsymbol{N} + \boldsymbol{r}(\nabla\cdot\boldsymbol{N})]\,\mathrm{d}v \qquad (7)$$

where S_1 is the surface enclosing V_1 and $\boldsymbol{n}$ is the outward unit normal vector on the surface element ds. In figure 1, V_1 and S_1 are shown as a full line. Then

$$\boldsymbol{f} = \frac{\mu_0 Q}{4\pi}\oint_{S_1}\frac{\boldsymbol{r}}{r^3}(\boldsymbol{J}\cdot\boldsymbol{n})(\boldsymbol{u}\cdot\boldsymbol{r})\,\mathrm{d}s. \qquad (8)$$

S_1 consists of a part of the outer surface S of the current distribution, on which $\boldsymbol{J}\cdot\boldsymbol{n}=0$, and the two surfaces Σ_1 and Σ_2 produced by sections of the distribution. The current loop is naturally complete, as indicated by the broken lines in figure 1, and there is flow of current through Σ_1 and Σ_2. The surface integral of equation (8) need only extend over Σ_1 and Σ_2 and

$$\boldsymbol{f} = \frac{\mu_0 Q}{4\pi}\left(\int_{\Sigma_1}\frac{\boldsymbol{r}}{r^3}(\boldsymbol{u}\cdot\boldsymbol{r})(\boldsymbol{J}\cdot\boldsymbol{n})\,\mathrm{d}s + \int_{\Sigma_2}\frac{\boldsymbol{r}}{r^3}(\boldsymbol{u}\cdot\boldsymbol{r})(\boldsymbol{J}\cdot\boldsymbol{n})\,\mathrm{d}s\right). \qquad (9)$$

If Σ_1 and Σ_2 coincide, completing the current loop, then since $\boldsymbol{n}$ has opposite directions on the two surfaces at every point, the integrals of equation (9) are equal and opposite and $\boldsymbol{f}=0$. This implies that, for a complete current distribution loop, the two laws predict the same force on both a volume current element and a part or the whole of another current distribution, when these lie outside the current distribution exerting the force.

A procedure will now be followed of incorporating Q to the current distribution. As a first step, Σ_1 and Σ_2 are allowed to touch each other except at points on the surface of a sphere of radius r and Q as its centre. Equation (9) then reduces to

$$f=\frac{\mu_0 Q}{4\pi}\oint_\Sigma \frac{\boldsymbol{r}}{r^3}(\boldsymbol{u}\cdot\boldsymbol{r})(\boldsymbol{J}\cdot\boldsymbol{n})\,\mathrm{d}s \tag{10}$$

where Σ is the surface of the sphere and $\boldsymbol{n}$ the unit vector normal to Σ and directed inwards, i.e. $\boldsymbol{n}=-\boldsymbol{r}/r$, as shown in figure 2. It must be emphasised that $\boldsymbol{J}$ is not discontinuous on Σ. The spherical surface is simply the region over which the surface integral is evaluated.

Taking a coordinate system with origin at Q and z axis in the direction of $\boldsymbol{u}$, as shown in figure 2, and using spherical coordinates (r, θ, φ) in which $\boldsymbol{J}\cdot\boldsymbol{n}=-J_r$, $\boldsymbol{u}\cdot\boldsymbol{r}=ur\cos\theta$ and $\mathrm{d}s=r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$, equation (10) becomes

$$f=\frac{\mu_0 Qu}{8\pi}r\oint_\Sigma \boldsymbol{n}J_r\sin 2\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{11}$$

Since J_r is finite on the sphere Σ, then the integral of equation (11) has a finite value and as r tends to zero, f also tends to zero, at least as fast as r does.

In fact, if J_r is expanded about O in a series involving x, y and z and derivatives of J_r, the integral of equation (10) is evaluated giving

$$f=-\frac{\mu_0 Q}{15}r^2[\nabla(\boldsymbol{u}\cdot\boldsymbol{J})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{J}]_{r=0}+\dots \tag{12}$$

where terms involving higher powers of r have been neglected. This shows that f tends to zero as r^2 does.

If the integrals in the expressions for the two forces are examined, it is seen from equations (1) and (2) that they involve terms of r^2 in the denominator. It is well known that volume integrals of the form $\int_V \mathrm{d}v/r^n$ converge if $n<3$ (e.g. Philips 1949), a fact

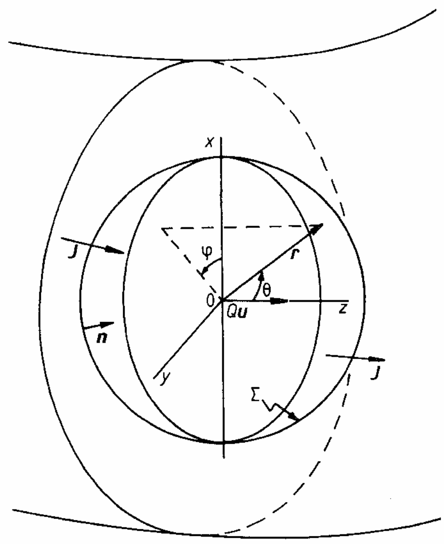

Figure 2. The sphere with centre at Q and radius r, defining the surface Σ over which the integral of equation (10) is evaluated.

which is widely used in electrostatics, magnetostatics, gravitation and field theory in general. To demonstrate the convergence of all the integrals involved, the volume integrals of all three different terms of equations (1) and (2) were evaluated for a sphere with centre at the origin O (and Q) and radius r, using an expansion of $\boldsymbol{J}$ about O in terms of x, y and z. The results are

$$\boldsymbol{K}_0=\int_V \frac{\boldsymbol{u}\times(\boldsymbol{J}\times\boldsymbol{r})}{r^3}\,\mathrm{d}v=-\frac{2\pi}{3}r^2[\boldsymbol{u}\times(\nabla\times\boldsymbol{J})]_{r=0}+\ldots \tag{13}$$

$$\boldsymbol{K}_1=\int_V \frac{\boldsymbol{r}}{r^3}(\boldsymbol{u}\cdot\boldsymbol{J})\,\mathrm{d}v=\frac{2\pi}{3}r^2[\nabla(\boldsymbol{u}\cdot\boldsymbol{J})]_{r=0}+\ldots \tag{14}$$

$$\boldsymbol{K}_2=\int_V \frac{\boldsymbol{J}}{r^3}(\boldsymbol{u}\cdot\boldsymbol{r})\,\mathrm{d}v=\frac{2\pi}{3}r^2[(\boldsymbol{u}\cdot\nabla)\boldsymbol{J}]_{r=0}+\ldots \tag{15}$$

$$\boldsymbol{K}_3=\int_V \frac{\boldsymbol{r}}{r^5}(\boldsymbol{u}\cdot\boldsymbol{r})(\boldsymbol{r}\cdot\boldsymbol{J})\,\mathrm{d}v=\frac{2\pi}{15}r^2[\nabla(\boldsymbol{u}\cdot\boldsymbol{J})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{J}]_{r=0}+\ldots \tag{16}$$

where terms involving higher powers of r are neglected.

The force on Q due to currents inside the sphere $(0, r)$ is, according to Ampère's law,

$$\begin{aligned}\boldsymbol{g}_{\mathrm{A}}&=\frac{\mu_0 Q}{4\pi}(2\boldsymbol{K}_1-3\boldsymbol{K}_3)\\&=\frac{\mu_0 Q}{30}r^2[7\nabla(\boldsymbol{u}\cdot\boldsymbol{J})-3(\boldsymbol{u}\cdot\nabla)\boldsymbol{J}]_{r=0}+\ldots\end{aligned} \tag{17}$$

and, according to the Biot–Savart law,

$$\begin{aligned}\boldsymbol{g}_{\mathrm{BS}}&=-\frac{\mu_0 Q}{4\pi}\boldsymbol{K}_0=\frac{\mu_0 Q}{4\pi}(\boldsymbol{K}_1-\boldsymbol{K}_2)\\&=\frac{\mu_0 Q}{6}r^2[\boldsymbol{u}\times(\nabla\times\boldsymbol{J})]_{r=0}+\ldots.\end{aligned} \tag{18}$$

The difference of the forces is

$$\begin{aligned}\boldsymbol{g}_{\mathrm{A}}-\boldsymbol{g}_{\mathrm{BS}}&=\frac{\mu_0 Q}{4\pi}(\boldsymbol{K}_1+\boldsymbol{K}_2-3\boldsymbol{K}_3)\\&=\frac{\mu_0 Q}{15}r^2[\nabla(\boldsymbol{u}\cdot\boldsymbol{J})+(\boldsymbol{u}\cdot\nabla)\boldsymbol{J}]_{r=0}+\ldots.\end{aligned} \tag{19}$$

As expected

$$\boldsymbol{g}_{\mathrm{A}}-\boldsymbol{g}_{\mathrm{BS}}=-(\boldsymbol{f}_{\mathrm{A}}-\boldsymbol{f}_{\mathrm{BS}})=-\boldsymbol{f} \tag{20}$$

since $\boldsymbol{f}$ is the force difference due to all the current distribution except the sphere $(0, r)$. It is seen that all the integrals tend to zero as r^2. The possibility of $\boldsymbol{f}$ being convergent as the difference of two diverging forces is thus also eliminated. It is concluded that there are no diverging integrals in the evaluation of the forces examined.

3. Discussion and conclusions

It has been shown that the force laws of both Biot–Savart and Ampère give the same value for the force exerted by a continuous complete current distribution $\boldsymbol{J}(\boldsymbol{r})$ on a volume current element $\boldsymbol{J}'\,dv'$, independent of whether this is outside or inside the current distribution. Since the Lorentz force of the Biot–Savart law is always normal to $\boldsymbol{J}$ for a volume current element, the equality of the two forces implies that the Ampère force is also normal to $\boldsymbol{J}$ at all points.

As the force on a single charge is influenced by other charges in its immediate neighbourhood, the use of both $Q\boldsymbol{u}$ and $\boldsymbol{J}$ as continuous functions of position is justified only if these quantities are taken as average values over volumes dv' large enough for the fluctuations in both the number of charges enclosed and their velocities to be adequately smoothed out. At the same time the variations of $\boldsymbol{J}$ or $Q\boldsymbol{u}$ over distances enclosed by dv' must be low enough so as not to invalidate the limiting procedures used in the mathematical analysis.

The equivalence of the two laws also implies that in the case of a complete macroscopic current distribution or a part of it, which are outside the current distribution exerting the force, this force is found to be the same by both laws. By integrating over a part of the current distribution it is found that the forces acting on each of the two parts are equal and opposite according to the Ampère law and therefore have to be equal and opposite according to the Biot–Savart law, which proves that a current distribution cannot exert a net force on itself and in fact obeys Newton's third law in this respect, according to both laws. Care must be taken in this case to take into account the whole of the distribution in the calculation of the force or the magnetic field at a given point when using the Biot–Savart law. No experiment can therefore be devised, able to differentiate between the two laws by measuring the force exerted on a part of a circuit by the circuit itself. Experimental evidence exists supporting this statement (Peoglos 1986).

The equality of the forces predicted by the two laws at each point of the current distribution shows that they also predict the same force distributions and stresses exerted by a current distribution on itself (or another current distribution), in disagreement with claims to the contrary.

The equivalence of the two laws is therefore considered to be complete regarding physically realisable current configurations in magnetostatics.

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